Chapter 13

PLASMA STABILITY THEORY *

INTRODUCTION

STABLE AND UNSTABLE EQUILIBRIA

13.1. Nearly all the proposals for confining plasmas at high temperatures, with a view to the realization of thermonuclear reactions, involve the use of magnetic fields in one form or another. Hence, the problem of the stability of the plasma in a magnetic field is of paramount importance. Early experimental and theoretical work showed that the plasmas in simple pinch and stellarator systems were basically unstable. Consequently, efforts have been made toward the development of a theory of plasma stability in the hope that the instabilities might be understood and methods developed for eliminating them or at least inhibiting their rate of growth.

13.2. Since the whole field of plasma confinement, and particularly of stability during confinement, is relatively new, and since there are undoubtedly several fundamentally different types of instability, no really satisfactory theory of plasma stability exists at the present time. Some progress has been made in connection with hydromagnetic instabilities, in which the plasma is treated as a simple hydrodynamic fluid interacting with a magnetic field, and a few significant conclusions capable of experimental test have been reached. However, it must be admitted that, except for one special case, the requirements for the stabilization of a plasma under conditions of practical interest have not yet been elucidated. Because the stability problem is basic to all aspects of research in controlled thermonuclear reactions, various methods of solving it which have been tried will be outlined and their limitations discussed. Although the treatments are highly mathematical, emphasis will be placed as far as possible on their physical significance.

13.3. In an unperturbed plasma in a state of equilibrium and at rest, the isotropic (scalar) pressure balance equation is

$$\nabla p = \frac{1}{c}(\mathbf{j} \times \mathbf{B}),$$

* Based on a draft prepared by Bergen R. Suydam.

which has three components corresponding to the three directions of the vectors. However, these three component equations relate four functions, namely, the pressure and the three components of the magnetic field. Consequently, equilibrium can be realized, in principle, with a wide variety of magnetic fields. The condition required is that $\nabla \times (\mathbf{B} \times \nabla \times \mathbf{B}) = 0$; for such fields, and only for such, is confinement of a scalar pressure possible. The central problem of plasma confinement is to determine which of the equilibria, of the many possible, are stable. In a stable system, a small perturbation will lead to an oscillation about the equilibrium state, but if the system is unstable, the perturbation will grow indefinitely. It will be seen that the treatment of the stability of a plasma confined by a magnetic field has many features in common with that of wave propagation. The question that will be asked is: Do perturbations remain at the same level or do they continue to grow?

13.4. Before discussing the various approaches to the subject of hydromagnetic plasma stability, it should again be emphasized that no really satisfactory theory has yet been developed. The present chapter should therefore be regarded as being in the nature of a progress report rather than a definitive account; it is included in this book because of the essential importance of the problem with which it is concerned.

SIMPLE EQUILIBRIUM SYSTEMS

13.5. A simple situation, for which it appears that an unequivocal solution to the stability problem is possible, may be treated in the following manner. Consider a portion of plasma having a uniform kinetic pressure p_0. The plasma is assumed to be perfectly diamagnetic, so that it contains no internal magnetic field. Suppose this plasma is confined by an external field, the value of which is B_0 at the surface of the plasma. The condition of equilibrium is then

$$p_0 = \frac{B_0^2}{8\pi},$$

so that the (uniform) pressure of the plasma is exactly balanced by the magnetic pressure of the confining field.

13.6. In order to determine whether this equilibrium is stable or not, imagine the surface of the plasma to be disturbed slightly by forming a wave (or "flute") with its crest and trough running parallel to the magnetic field lines. If the trough and crest have the proper dimensions, there will be no change in the volume of the plasma and hence no change in the pressure. Suppose, in the first place, that the confining field lines are curved so that they are concave toward the plasma. This is represented in Fig. 13.1A, where I shows the side view and II the end view, in which the wavelike disturbance is indicated. It follows from Maxwell's equation $\nabla \times \mathbf{B} = 0$ that the magnetic field strength will diminish with increasing distance from the center of curvature of the field lines. Consequently, in the case under consideration, the magnetic field on the crest of the wave is such that $B^2 < B_0^2$, whereas in the trough $B^2 > B_0^2$. As a result, a net force acts in such a sense as to heighten the crest and deepen the trough, that is to say, the disturbance will tend to grow in amplitude. It follows, therefore, that the particular equilibrium is unstable. On the other hand, if the magnetic field lines are convex toward the plasma, as represented in Fig. 13.1B, the net magnetic force is such as to depress the crest and raise the

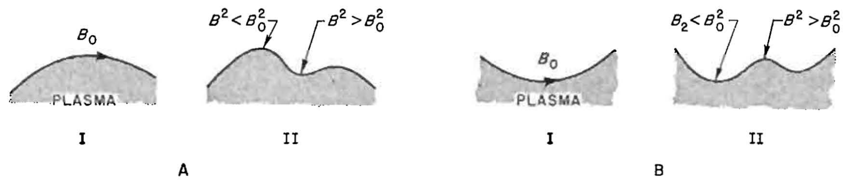

FIG. 13.1. Unstable and stable magnetic field-plasma configurations.

trough of the disturbance under consideration. In other words, the force acts in such a manner as to counteract the disturbance and the equilibrium is stable.

13.7. The foregoing discussion suggests and rigorous analysis (see Appendix A) confirms that a perfectly diamagnetic plasma, i.e., one with no internal magnetic field, is stable or not according to the curvature of the lines of the confining field at its surface [1-3]. If every such line is *everywhere* convex toward the plasma or, in other words, if the center of curvature of the magnetic lines on the plasma-field interface always lies in the direction away from the plasma, then the system is stable. On the other hand, if the lines are *anywhere* concave toward the plasma, so that the center of curvature is within (or toward) the plasma, then the system must be unstable. Thus, a pinched discharge consisting of a very thin sheath of essentially infinite conductivity, with no trapped axial field, will be fundamentally unstable.

13.8. In order for a completely diamagnetic plasma to be stable when confined by a magnetic field, it cannot form a convex body, but must have cusps. A number of different configurations satisfying this requirement, some of which are being studied experimentally, were described in §11.22 *et seq.*

HYDROMAGNETIC THEORY

NORMAL MODE ANALYSIS

13.9. The foregoing discussion has been essentially qualitative in nature and has dealt with the restricted situation of a perfectly diamagnetic plasma. For practical purposes, information is required concerning the rate of growth of various instabilities that might develop and also on how the growth may be inhibited. Furthermore, in many cases of interest, there is a magnetic field inside as well as outside the plasma; the situation is then more realistic, but also more complicated, than that considered above. The displacement of

the plasma surface now results not only in a shift of the magnetic lines in the vacuum field outside, but will also cause the lines within the plasma to bend, stretch, or be distorted in other ways. This leads to additional forces which require more detailed analysis. The mathematical treatment of such a system is very complex and so it is necessary to make approximations in the model of the plasma in order to simplify the results to the point where they can be applied to a practical geometry.

13.10. The simplest theory to be described here is based on the treatment of a plasma as a simple (ideal) hydrodynamic fluid which is subjected to the action of electromagnetic forces. In other words, it involves a combination of the standard equations of hydrodynamics with Maxwell's equations of electromagnetism, leading to a complete set of differential equations determining the motion of the plasma. This hydromagnetic theory is of interest both because it has led to concrete results and because the domain of its validity has been shown to be much wider than might have been anticipated from the approximations made in its development (§13.46). Essentially, the theory of plasma stability has features in common with that of plasma waves, since instabilities may be regarded merely as being waves which grow at an ever-increasing rate.

13.11. In order to apply the hydromagnetic equations to a plasma, a number of simplifying postulates are made. First, collisions between particles are assumed to occur so frequently that the velocity distribution of the particles is isotropic, or very nearly so. Actually, this means that the plasma can be characterized at each point by a mass density ρ, a scalar pressure p, and a single temperature. Incidentally, the frequent collisions provide a mechanism for keeping the neighboring particles together so that they form a coherent element of fluid which can be treated by simple hydrodynamics. The second postulate is that, as a result of the high rate of particle collisions, heat flow is inhibited within the plasma, so that the system is adiabatic. Finally, it is supposed that the plasma temperature is so high that it is essentially a perfect conductor. This means that such currents as flow produce no ohmic heating nor do they require electric fields to drive them.

13.12. With these simplifications, the plasma behaves like the perfect fluid of ordinary hydrodynamics, except that it is subject to electromagnetic forces. Exactly as in fluid mechanics, there are three conservation equations, as follows:

Conservation of Mass

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0, \tag{13.1}$$

where d/dt represents the hydrodynamic operator, i.e.,

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla,$$

$\mathbf{v}$ being the vector velocity of the plasma and t the time. Equation (13.1) is also known as the equation of continuity.

Conservation of Momentum

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c}(\mathbf{j} \times \mathbf{B}), \tag{13.2}$$

where d/dt is the hydrodynamic operator, and $\mathbf{j}$ and $\mathbf{B}$ are, as before, the current density and magnetic field vectors. Equation (13.2), which neglects the effect of gravity and of the electric field, since the latter is assumed to be zero, is sometimes known as the force (or motion) equation (cf. §4.116). It describes the motion of the plasma under the combined action of a pressure gradient and a magnetic body force.

Conservation of Energy

$$\frac{dp}{dt} + \gamma p(\nabla \cdot \mathbf{v}) = 0, \tag{13.3}$$

where d/dt is the hydrodynamic operator defined above and γ is the ratio of the specific heats, which may generally be taken as the monatomic gas value, i.e., 5/3, for a plasma. This equation implies adiabatic behavior, as may be seen by combining it with equation (13.1) to give

$$\frac{1}{p} \cdot \frac{dp}{dt} = \frac{\gamma}{\rho} \cdot \frac{d\rho}{dt}$$

and integrating; the result is

$$p = \text{constant} \times \rho^{\gamma},$$

which is the familiar adiabatic law for a gas.

13.13. In addition to the three hydrodynamic equations given above, there are the Maxwell equations. As the plasma is assumed to be perfectly conducting, an observer moving with the local velocity of the plasma would experience no electric field; hence,

$$\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B}) = 0.$$

When this is set into Maxwell's equation

$$\nabla \times \mathbf{E} = -\frac{1}{c} \cdot \frac{\partial \mathbf{B}}{\partial t},$$

there is obtained

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{13.4}$$

as the equation of motion of the magnetic field. Physically, this result implies that the plasma and the magnetic field move together.

13.14. Use will also be made of the Maxwell relationship, neglecting the displacement current (cf. §3.11),

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}, \tag{13.5}$$

which is essentially a definition of the current density. Finally, the particles move in such a way that the charge density σ is given by

$$\nabla \cdot \mathbf{E} = 4\pi\sigma,$$

and the divergence of the magnetic field is zero, i.e.,

$$\nabla \cdot \mathbf{B} = 0.$$

13.15. Consider an equilibrium system at rest, so that $\mathbf{v} = 0$, characterized by the state variables ρ_0, p_0, and $\mathbf{B}_0$. These variables are connected by equation (13.2) which now takes the form

$$0 = -\nabla p_0 + \frac{1}{c}(\mathbf{j}_0 \times \mathbf{B}_0).$$

Suppose that the state of equilibrium is perturbed by a very small amount, so that the pressure becomes $p_0 + p'$, the magnetic field becomes $\mathbf{B}_0 + \mathbf{B}'$, and so on. The system will now have a small velocity $\mathbf{v}$, and the quantities p', $\mathbf{B}'$, and $\mathbf{v}$ will develop with time in accordance with the equations of motion given above. It is essential that the perturbations p', $\mathbf{B}'$, etc., are considered to be small, so that their products, both with themselves and with each other, are negligible. In this way, it is possible to achieve *linear* equations of motion.

13.16. It is convenient to introduce a displacement variable vector $\boldsymbol{\xi}$ defined by

$$\frac{\partial \boldsymbol{\xi}}{\partial t} \equiv \mathbf{v} \quad \text{or} \quad \boldsymbol{\xi} \equiv \int \mathbf{v}\, dt,$$

so that equation (13.4) leads to

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0\right)$$

which, upon integration, yields

$$\mathbf{B}' = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0). \tag{13.6}$$

13.17. Writing $p_0 + p'$ for p in equation (13.3) and introducing the definition of $\boldsymbol{\xi}$, this expression takes the linearized form

$$\frac{\partial p'}{\partial t} + \left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right) \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right) = 0$$

and integration then gives

$$p' = -\gamma p_0 \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p_0. \tag{13.7}$$

From equation (13.2), with $\mathbf{j}$ equal to $\mathbf{j}_0 + \mathbf{j}'$, it follows that

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla p' + \frac{1}{c}\left[(\mathbf{j}' \times \mathbf{B}_0) + (\mathbf{j}_0 \times \mathbf{B}')\right].$$

Upon substituting equations (13.6) and (13.7) for $\mathbf{B}'$ and p', respectively, and utilizing equation (13.5), there results

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \nabla[\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0] + \frac{1}{4\pi}[\nabla \times \nabla \times (\xi \times \mathbf{B}_0) \times \mathbf{B}_0 + \nabla \times \mathbf{B}_0 \times \nabla \times (\xi \times \mathbf{B}_0)], \quad (13.8)$$

where ρ_0, p_0, and $\mathbf{B}_0$ are determined by the initial equilibrium distribution.

13.18. Because equation (13.8) is linear in the unknown vector function ξ, it can, without loss of generality, be simplified by analyzing it into its Fourier components. Thus, assuming the time dependence to be of the form

$$\xi(\mathbf{x}, t) = \xi(\mathbf{x}) \exp(i\Omega t),$$

where $\mathbf{x}$ represents the space coordinate vector, equation (13.8) becomes

$$-\rho_0 \Omega^2 \xi = \nabla[\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0] + \frac{1}{4\pi}[(\nabla \times \mathbf{Q}) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{Q}], \quad (13.9)$$

where $\mathbf{Q}$ is defined by

$$\mathbf{Q} \equiv \nabla \times (\xi \times \mathbf{B}_0).$$

It should be noted that the ξ in equation (13.9) and subsequently is really the spatial component $\xi(\mathbf{x})$, as is apparent from the presence of Ω^2 on the left side of the equation. The argument $(\mathbf{x})$ is omitted here to simplify the representation. Equation (13.9), called the *normal mode equation*, combined with the boundary conditions on ξ, namely, (*a*) ξ is finite everywhere, and (*b*) the normal component of ξ vanishes on any metallic conductors present, e.g., magnetic field coils or external conductor, forms an eigenvalue problem for the determination of Ω^2. If every eigenvalue is positive, then every mode is periodic in time and is a simple (traveling or standing) wave. If, on the other hand, one of the eigenvalues of Ω^2 is negative, then one of the two associated modes grows exponentially with time and the system is unstable.

13.19. The normal mode equation (13.9) is analogous to that used in the study of acoustical waves and is identical with that which is applicable to Alfvén (plane) waves in a plasma having a finite pressure. When the magnetic field is uniform, a further Fourier analysis, with respect to the space variable $\mathbf{x}$, is possible; thus, ξ may be taken to be of the form

$$\xi = (\xi_1, \xi_2, \xi_3) \exp(i\mathbf{k} \cdot \mathbf{x} + i\Omega t), \quad (13.10)$$

and then equation (13.9) reduces to a system of linear algebraic equations which determine the so-called dispersion relationship between Ω^2 and $\mathbf{k}$. In many cases, the situation is further simplified by taking p to be constant, so that ∇p is zero. An example of the application of the normal mode method to the stability of a pinched discharge is given in Appendix B to this chapter.

VARIATIONAL METHOD

13.20. Although solution of the normal mode equation is possible in the simpler cases, with more complicated geometries equation (13.9) remains as a system of three coupled partial differential equations and is quite unmanageable. In these circumstances, another procedure can be used which replaces the normal mode problem of equation (13.9) by a simple variational problem [4]. By sacrificing detailed knowledge of the normal modes, it is possible, in the following manner, to determine merely whether a given system is stable or not. First, as a result of scalar multiplication of equation (13.9) by ξ followed by integration over the volume of the plasma, there is obtained

$$\Omega^2 \int \rho(\xi \cdot \xi)\, d^3x = \int \xi \cdot F\xi\, d^3x, \quad (13.11)$$

where F represents the negative of the complicated differential operator (on ξ) on the right side of equation (13.9). Because the normal component of ξ vanishes on the boundary, i.e., at infinity or at a conductor, the right side of equation (13.11) can be simplified by partial integration; thus,

$$\Omega^2 = \frac{\delta W}{\int \rho(\xi \cdot \xi)\, d^3x}, \quad (13.12)$$

where δW is defined by

$$\delta W \equiv \int \xi \cdot F\xi\, d^3x$$

$$= \int \left[\frac{1}{4\pi}(\mathbf{Q} \cdot \mathbf{Q} - \nabla \times \mathbf{B} \cdot \mathbf{Q} \times \xi) + \gamma p(\nabla \cdot \xi)^2 + (\nabla \cdot \xi)(\xi \cdot \nabla p)\right] d^3x. \quad (13.13)$$

13.21. The operator F can be shown to be self-adjoint, and so it follows that any vector function ξ which makes equation (13.12) stationary is an eigenmode and vice versa; furthermore, this stationary value of Ω^2 is an eigenvalue. Thus, the eigenvalue problem is exactly equivalent to minimizing Ω^2 as given by equation (13.12). As seen earlier, stability of the plasma system to a perturbation depends upon whether Ω^2 is always positive (stable) or whether any of the Ω^2 values can be negative (unstable). But the denominator of equation (13.12) is positive, and so the sign of Ω^2 depends only on that of the numerator, i.e., δW. Thus, the normal mode stability condition can be replaced by the following variational principle: the necessary and sufficient condition that the system be stable is that δW, as given by equation (13.13), be not negative for every ξ which satisfies the boundary conditions.

13.22. The variational principle developed above is also called an energy principle, for it turns out, as might have been expected, that δW is equal to the change in potential energy of the system resulting from the displacement ξ. Consequently, only those equilibrium systems are stable in which every conceivable small perturbation is accompanied by an increase of potential energy, i.e., by doing work on the system.

13.23. The energy principle described above has been used to provide a rigorous confirmation of the general arguments presented earlier which indicate that stable confinement of a perfectly diamagnetic plasma by a magnetic field is possible only when the center of curvature of the lines of force always lies on the field side of the plasma-field boundary. The details of the argument are given in Appendix A to this chapter.

APPLICATIONS OF STABILITY THEORY

STABILITY OF THE PINCHED DISCHARGE

13.24. An important application of stability theory has been in the study of the straight cylindrical pinched discharge [5]. Here the simple nature of the geometry permits some generality in the consideration of pressure distributions. In the earliest models, the plasma was treated as a cylindrical rod in which the current was confined to an infinitely thin layer at the surface (cf. §7.35). As ∇p_0 and $\mathbf{j}_0$ vanish, the normal mode equation (13.9) then reduces to the simpler form

$$-\rho\Omega^2\xi = \nabla(\gamma p_0 \nabla \cdot \xi) - \frac{1}{4\pi}[\mathbf{B}_0 \times (\nabla \times \mathbf{Q})]. \tag{13.14}$$

The spatial component of the displacement variable vector ξ or, in fact, of any perturbation represented by the primed quantities given above, may be written in the form of equation (13.10), so that for cylindrical geometry

$$\xi = \xi(r) \exp(im\theta + ikz), \tag{13.15}$$

where m is an integer, assumed to be positive, and k is real. As seen in §7.34, the value of m determines the nature of the plasma perturbation, thus, $m = 0$ represents a necking-off, $m = 1$ is a kink or spiral, etc. The wave length of the perturbation is equal to $2\pi/|k|$. If equation (13.15) is substituted into equation (13.14), there is obtained a set of three coupled, differential equations in the three space coordinates. Upon applying the boundary conditions appropriate to the problem, namely, that (*a*) the magnetic field is tangent to the plasma-field interface and (*b*) the total pressure $p + B^2/8\pi$ is continuous across the interface, the required dispersion relation can be found. From this the condition for stability, i.e., Ω^2 is always positive, is derived. Except for the simplest cases, the actual determination of which configurations are stable and which are not, for different values of m, involves laborious numerical calculations.

13.25. The first application of the normal mode analysis to a pinched discharge was made for a cylindrical plasma in which the current was confined to an extremely thin sheath at the surface, and there was no axial magnetic field [6]. For the case of $m = 1$, it was found that the system was unstable. The calculations were extended to other values of m and it was shown that instability would also arise from $m = 0$, and, to a lesser extent, for $m \geqslant 2$. For sufficiently short wave lengths, the growth of the unstable perturbations, for all m values, was found to have an e-folding time of $(2|k|p_0/\rho_0 r_0)^{1/2}$, where r_0 is the tube radius [7]. Since the velocity of sound is equal to $(\gamma p_0/\rho_0)^{1/2}$, it has become the common practice to state that the instabilities in a pinched discharge grow, roughly, with a velocity equal to that of sound in the given plasma. At high temperatures, this can be very large.

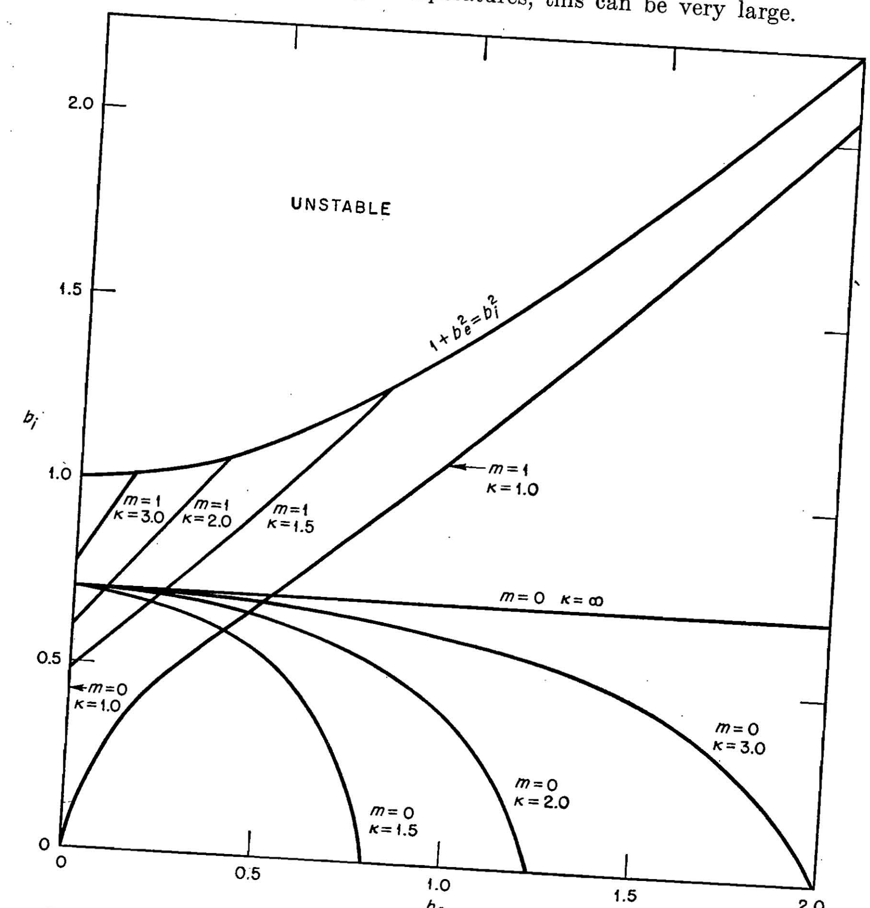

FIG. 13.2. Conditions of stability to $m = 0$ and $m = 1$ perturbations in pinched discharge.

13.26. The normal mode method has also been applied to the case in which there is an axial magnetic field inside as well as outside the plasma, and also with conducting walls surrounding the system [8-10]. The stability conditions derived in this manner were found to be essentially identical with those obtained by a procedure similar in principle to the variational method described above [11].

13.27. The results of the computations are summarized in Fig. 13.2, for the two cases $m = 0$ and $m = 1$; the quantities b_i and b_e, defined by

$$b_i \equiv \frac{B_z\,(\text{int})}{B_\theta} \quad \text{and} \quad b_e \equiv \frac{B_z\,(\text{ext})}{B_\theta},$$

represent the ratios of the B_z field in the interior and exterior, respectively, of the plasma to the B_θ field at the exterior surface of the constricted discharge [8]. Since the pinched plasma can exist only if the magnetic pressure due to the external fields B_θ and B_z (ext) is greater than or equal to that due to the internal field B_z (int), i.e.,

$$\frac{B_\theta^2}{8\pi} + \frac{B_z^2\,(\text{ext})}{8\pi} \geqslant \frac{B_z^2\,(\text{int})}{8\pi},$$

it follows that the condition can also be expressed by

$$1 + b_e^2 \geqslant b_i^2.$$

Hence, in Fig. 13.2, the curve marked $1 + b_e^2 = b_i^2$ represents the limit imposed by equilibrium. The parameter κ is equal to r_0/r, as defined in §7.17, where r_0 is the radius of the conducting wall, which may be taken as essentially the same as that of the tube containing the gas. A pinched discharge should be stable within the area between the curve for the given m and κ values and the limiting curve for $1 + b_e^2 = b_i^2$. Complete stability for a specified κ occurs when the system lies within the stability region for both $m = 0$ and $m = 1$.

13.28. It will be observed that as the pinch ratio κ increases, the stability region becomes steadily smaller, so that stability requires decreasing values of b_e, i.e., of the axial magnetic field external to the pinched discharge. In fact, when $\kappa = 5$, stability is possible only if b_e is zero, so that there is no external axial field. If the pinch ratio exceeds 5, the calculations indicate that stability cannot be achieved by means of an axial magnetic field and a conducting wall. Since it is desirable to have the maximum possible compression, i.e., maximum κ, it is evidently necessary to try to keep the axial field trapped completely within the plasma. The stability conditions for this situation were represented in Fig. 7.7 and discussed in §7.35 *et seq.*

13.29. In later work on pinch stability, more realistic cases, in which the current is not confined to an infinitely thin layer of plasma, have been investigated. By allowing the current layer to have structure, more complicated plasma modes become possible and the requirements for stability become correspondingly more severe. The general nature of the results given above remains unchanged, but an analysis now indicates that stability is not possible if there is an axial magnetic field outside the plasma in the same direction as the field within the plasma. It appears that, when the current layer is thin but finite, an axial field in the vacuum outside the plasma is required for stability but it must be in a sense *opposite* to that inside the plasma. However, the reversed field alone is not sufficient for stability, since a special surface distribution is required in addition [12].

13.30. In the case of very thick current layers, it is a relatively simple matter to utilize the variational principle to derive a necessary, although not sufficient, condition for stability which relates the pressure gradient in the plasma to the torsion of the magnetic field lines [13]. The general conclusion was given in §7.97, and will be repeated here for completeness. The lines of the combined axial and azimuthal fields form spirals and their pitch, μ, may be defined by

$$\mu \equiv \frac{B_\theta}{rB_z},$$

where r is the radius of the pinched discharge; the quantity $(1/\mu)(\partial\mu/\partial r)$ measures the torsion (or "shear") of the field lines. The necessary stability condition alluded to above is then

$$\frac{r}{4}\left(\frac{1}{\mu}\cdot\frac{\partial\mu}{\partial r}\right)^2 + \frac{8\pi}{B_z^2}\left(\frac{\partial p}{\partial r}\right) > 0.$$

As seen in §7.99, one possible way of realizing this condition is to apply an axial magnetic field outside the plasma in a direction opposite to that of the field within the plasma.

13.31. It may be mentioned that stability conditions for the pinched discharge, which are both necessary and sufficient, have been derived [5, 14, 15]. However, these involve complex mathematical expressions and their application requires lengthy calculations. It would appear to be almost as simple to perform the variational problem directly for each case with the aid of a computing machine.

STABILITY IN STELLARATOR SYSTEMS

13.32. As indicated in Chapter 8, some use of hydromagnetic stability theory has been made in connection with stellarator systems, particularly of the kink ($m = 1$) and interchange instabilities. The situation in the stellarator differs from that of the pinched discharge in the respect that there is essentially only one type of magnetic field, namely the axial field, in the former case and this is largely within the plasma. Any azimuthal field that arises from current passing through the plasma, e.g., for ohmic heating, is negligible.

13.33. The normal mode treatment has been applied to the situation while the plasma is being heated ohmically [16]. It has been found that, in a stellarator in which the rotational transform arises from the large-scale geometry of the tube rather than from small local perturbations of the magnetic field produced by helical windings, the $m = 1$ mode is the only one which can cause serious instability. This occurs when the rotational transform due to the longitudinal current just compensates for that caused by the geometry of the stellarator tube. The theoretical treatment indicates that, in this case, the effect of an external conductor is negligible, although such a conductor tends to suppress the kink instability in a pinched discharge (§7.27).

13.34. By means of the variational principle, it has been shown that, when the rotational transform arises from the geometry of the stellarator, as considered above, the plasma is always subject to the interchange (or flute) instability. Utilizing the same procedure it has been found, however, that this can be stabilized by means of a transverse, helical magnetic field, which can itself produce a rotational transform, as explained in Chapter 8 [17]. Consider a plasma with an axial magnetic field inside it; this field must be so disposed that any conceivable displacement requires work to be done. The plasma may be visualized as a set of "onionskin" layers, each layer being a surface made up of magnetic field lines. If now the field lines are given torsion (or shear), that is, if the directions of the lines are always different in adjacent layers, then no displacement of the plasma will be possible which does not bend the lines. Even if there is a wavelike displacement in which the crests and troughs are aligned with the magnetic field in one particular layer, the torsion of the field line will spoil the alignment in neighboring layers. In such a system work must be done on the magnetic field for any conceivable displacement, and stability will be determined by whether the heat energy of the plasma can supply this work or not.

13.35. With the aid of the variational (or energy) principle it has been found that torsion of the magnetic field lines, such as would be produced by means of a transverse, helical field, as described in §8.19, has a stabilizing effect. The system is stable against interchange so long as the heat energy available is not too large or, in other words, provided the plasma pressure is small. These limiting pressures appear to be so low as to require β values of the order of 0.01. Furthermore, provided the rotational transform angle produced by the helical field is greater than and has the same sign as that caused by the ohmic heating current, it is expected that the plasma will be stable to perturbations of all m values.

COLLISIONLESS (NONEQUILIBRIUM) THEORIES

THE BOLTZMANN EQUATION

13.36. Although the hydromagnetic model of a plasma, based on the hydrodynamic equations (13.1), and (13.2), and (13.3), has been treated at some length, it is not because it is by any means a final or definitive theory. It is rather because it has led to results which have provided a significant insight into the stability of a plasma confined by a magnetic field. There are several defects in the theory, and so efforts are being made to develop other theories which are more rigorous but, on the other hand, more complicated; some of these alternative models will be discussed below.

13.37. The main objection to the hydrodynamic theory is concerned with the postulate of frequent collisions. As the temperature increases, the cross sections for Coulomb scattering collisions decrease (§4.48 *et seq.*) and at temperatures of thermonuclear interest they become so small that the mean free path of a particle may be comparable with or, depending on the density, may even be considerably larger than the dimensions of the containing vessel. Thus, although collisions may be frequent at the densities and temperatures of most laboratory experiments, they certainly will be much less common in an actual thermonuclear reactor. The absence of collisions has several important consequences.

13.38. In the first place, there is now no mechanism for partitioning the kinetic energy of the particles equally among their three degrees of freedom; thus, $p_{\parallel}$, the pressure parallel to the magnetic field lines, will not necessarily be the same as $p_{\perp}$, the pressure perpendicular to the field. Second, there is nothing to inhibit the motion of particles along the lines of force, and so heat flow parallel to these lines is likely to be very rapid. Third, in a reacting plasma, at least, electrons and ions will undoubtedly be at different temperatures. In brief, therefore, it may be concluded that the plasma will not be even in local thermodynamic equilibrium. If by any chance it is, then this local equilibrium will be upset if the plasma is in any way disturbed. It should be pointed out, too, that electrons and ions do not have to move together, and local deviations of charge density from zero may occur.

13.39. For the foregoing reasons, the nonequilibrium alternatives to the hydromagnetic theory start from the opposite assumption to that in §13.11, namely, it is supposed that there are no collisions or, if there are any, their effects are neglected. This postulate alone is not enough to bring about a sufficient simplification of the problem and another assumption is needed. The one made is to say that the magnetic field is so strong that the radii and periods of the gyromagnetic orbits of all particles are much smaller than any distances and times, respectively, of interest. In place of the three hydrodynamic equations (§13.12), the nonequilibrium theories either utilize directly a form of the Boltzmann distribution equation applicable to charged particles in which the collision term is disregarded [19-21], or they take averages over individual orbits which lead to results essentially equivalent to the Boltzmann equation [22, 23].

13.40. The so-called collisionless Boltzmann equation* may be written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})\right] \cdot \nabla_v f = 0, \qquad (13.16)$$

*Some workers in the field refer to this as the Liouville equation, since the latter has no collision term. However, it seems preferable to use the expression "collisionless Boltzmann equation" to imply that the Boltzmann equation is "correct," but collisions are being ignored as an approximation in order to simplify the treatment.

where the distribution function f, which is really $f(\mathbf{x}, \mathbf{v}, t)$ is defined so that $f\delta^3\mathbf{x}\delta^3\mathbf{v}$ is the number of particles in the volume element $\delta^3\mathbf{x}$ and in the velocity range $\delta^3\mathbf{v}$ at position $\mathbf{x}$ and velocity $\mathbf{v}$ in the six-dimensional (position-velocity) phase space at time t; the symbols e and m represent the charge and mass of the particles and ∇_v is the gradient operator for differentiation with respect to the components of the velocity vector rather than with respect to the space coordinates [24, 25].

13.41. In order to obtain equations for the mass and momentum balance, the zero- and first-order moments of the Boltzmann equation are derived by multiplying equation (13.16) by $m d\mathbf{v}$ and $m\mathbf{v}d\mathbf{v}$, respectively, and integrating over all velocity space. Utilizing the Maxwell relationships, as in §13.13, the results are found to resemble formally equations (13.1) and (13.2) of hydrodynamics, except that the gradient of a scalar pressure, i.e., ∇p, in the latter is replaced by the divergence of a pressure tensor.

13.42. A difficulty arises in connection with the development of the second-order moment of equation (13.16) to yield an expression for the energy balance. This introduces third moments of the distribution function which are the components of the heat flow vector. In ordinary hydrodynamics, use is made of the fact that, as a result of collisions, the distribution function is Maxwellian (or nearly so) in form. The third moments can then be related to the second moments and the equations can be closed. In addition, the heat flow is often so small, because of frequent collisions, that it can be neglected; in this event, the result reduces to the same form as the adiabatic equation (13.3).

13.43. In applying the situation to a plasma, however, the neglect of collisions makes it impossible to obtain closed equations because the heat flow terms cannot be evaluated. The simplest way of dealing with the situation is arbitrarily to cut off the energy balance equation at the third moment, which is equivalent to assuming no heat flow. The equations now form a closed system. The postulate of a strong magnetic field (§13.39) means that it is necessary to distinguish only between the components $p_{\parallel}$ and $p_{\perp}$ of the pressure tensor. As a result, the single adiabatic law, which can be written as $d(p/\rho^{\gamma})/dt = 0$, is replaced by

$$\frac{d}{dt}\left(\frac{p_{\parallel}B^2}{\rho^3}\right) = 0$$

and

$$\frac{d}{dt}\left(\frac{p_{\perp}}{\rho B}\right) = 0,$$

where the d/dt is the hydrodynamic operator defined in §13.12.

13.44. The nonequilibrium theory outlined above has a serious defect, since it artificially suppresses heat flow under collisionless conditions when thermal transport along the magnetic lines is expected to be considerable.

Nevertheless, the theory has led to some new results of interest. One of these, which is typical of nonequilibrium theories, arises from a consideration of the simple case of a plasma having initially uniform pressures $p_{\parallel}$ and $p_{\perp}$ and density ρ_0 in a uniform field $\mathbf{B}_0$. By solving the problem of plane waves, it is found that exponentially growing waves are possible. In other words, instability may occur even in a plasma with a uniform distribution. The condition for stability is

$$\frac{p_{\perp}^2}{6(p_{\perp} + B^2/8\pi)} < p_{\parallel} < p_{\perp} + \frac{B^2}{4\pi},$$

where $p_{\perp}$ and $p_{\parallel}$ are the respective undisturbed values; hence, for stability of the plasma, $p_{\parallel}$ must lie within the range indicated [26]. A slightly more restrictive stability condition, which takes heat flow into consideration, may be written in the analogous form [27]

$$\frac{p_{\perp}^2}{p_{\perp} + B^2/8\pi} < p_{\parallel} < p_{\perp} + \frac{B^2}{4\pi}.$$

13.45. An interesting new general result derived from a more elaborate nonequilibrium theory is that the Maxwellian velocity distribution leads to the most stable state and that deviations from this distribution, if sufficiently great, can in themselves cause instability. A particular case arises when $p_{\parallel}$ and $p_{\perp}$ are unequal.

13.46. In addition to the foregoing conclusions, some other interesting features of nonequilibrium theories will be noted. First, it may be mentioned that it has been found possible to develop an energy principle, analogous to that described in §13.21, based directly on the collisionless Boltzmann equation [28]. An essentially equivalent principle has also been derived by summing over the orbits of individual particles [23]. These theories lead to the important result that, if $p_{\parallel}$ and $p_{\perp}$ are initially equal, then the δW predicted is at least as large as that obtained from the simple hydromagnetic theory, as given by equation (13.13). This means that, when stability is predicted on the basis of the energy (or variational) principle in hydromagnetic theory, the conclusion can be accepted with confidence in many cases. The utility of the simple theory would thus appear to be greater than its numerous deficiencies would suggest.

13.47. All three energy principles mentioned so far are applicable only to states of static equilibrium of the system as a whole. If the plasma is in a steady state of motion, e.g., if it is rotating, the kinetic energy of this motion is an additional source from which an instability might be driven. What this means mathematically is that, in linearizing equation (13.2), or its equivalent, it is not permissible to neglect terms arising from $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$.

13.48. A method has been developed for handling the problem arising when the plasma is in motion, by assuming a strong magnetic field and no collisions

[20, 29, 30]. The results lead to linearized equations of motion, i.e., to normal mode equations, rather than to a variational principle. This means that the treatment is very difficult to apply to anything but the simplest geometry, although it may be amenable to machine calculations in other cases.

APPENDIX A: APPLICATION OF THE VARIATIONAL PRINCIPLE TO A PERFECTLY DIAMAGNETIC PLASMA

13.49. In the following treatment it will be shown how the variation principle (or δW-formalism) has been applied to determine how the stability of a perfectly diamagnetic plasma confined by a magnetic field depends on the direction of the curvature of the vectors of the field lines [4]. Starting with the basic equation (13.12), with δW defined by equation (13.13), some general results will first be derived, and then they will be applied to the case of immediate interest.

13.50. The integral for δW can be transformed by partial integration utilizing the relationships

$$-\boldsymbol{\xi}\cdot\nabla\times\mathbf{Q}\times\mathbf{B}_0 = Q^2 + \nabla\cdot[\mathbf{Q}\times(\boldsymbol{\xi}\times\mathbf{B}_0)]$$

and

$$-\boldsymbol{\xi}\cdot\nabla[\gamma p\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p_0] = \gamma p_0(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\nabla p_0) - \nabla\cdot[\gamma p_0\boldsymbol{\xi}\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}(\boldsymbol{\xi}\cdot\nabla p_0)].$$

It follows, therefore, that

$$\delta W = \delta W_f - \frac{1}{2}\int_S (n\cdot\boldsymbol{\xi})[\gamma p_0\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p_0 - \frac{1}{4\pi}\mathbf{B}_0\cdot\mathbf{Q}]\,ds \qquad (13.17)$$

where δW_f is defined by

$$\delta W_f \equiv \int_V \left[\frac{Q^2}{4\pi} - \mathbf{j}\cdot\mathbf{Q}\times\boldsymbol{\xi} + \gamma p_0(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\nabla p)\right]d^3x, \qquad (13.18)$$

and ds is an element of surface; the integral in equation (13.17) is taken over the surface of the plasma whereas that in equation (13.18), as in equation (13.13), is over the volume.

13.51. In normal laboratory experiments, the "vacuum" outside the plasma usually contains enough particles to conduct current, even if it will not produce a pressure; hence, the plasma boundary lies on the conducting walls, where $(\mathbf{n}\cdot\boldsymbol{\xi}) = 0$, the symbol $\mathbf{n}$ representing the unit vector normal to the surface; the surface integral then contributes nothing. In idealized cases, however, where a perfect vacuum exists or when the walls are nonconducting, the surface integral must be evaluated. This is best done by using boundary conditions to simplify the integral.

13.52. The first step is to calculate the vacuum fields; these are indicated by a circumflex. The electric field is given by

$$\hat{\mathbf{E}} = \hat{\mathbf{E}}_0 + \hat{\mathbf{E}}' = \hat{\mathbf{E}}',$$

since $\hat{\mathbf{E}}_0 = 0$, and the magnetic field is

$$\hat{\mathbf{B}} = \hat{\mathbf{B}}_0 + \hat{\mathbf{B}}'.$$

The perturbations $\hat{\mathbf{E}}'$, $\hat{\mathbf{B}}'$ together form an electromagnetic field and are therefore derivable from a potential. Thus, if $\mathbf{A}$ represents a vector potential, then

$$\hat{\mathbf{B}}' = \nabla\times\mathbf{A} \qquad (13.19)$$

and

$$\hat{\mathbf{E}}' = -\frac{1}{c}\cdot\frac{\partial\mathbf{A}}{\partial t} = -\frac{i\Omega}{c}\mathbf{A}, \qquad (13.20)$$

and $\mathbf{A}$ satisfies the equation

$$\nabla\times\nabla\times\mathbf{A} = 0. \qquad (13.21)$$

No scalar potential is necessary as no charge accumulations are assumed.

13.53. Next, the boundary conditions will be derived. In the first place, the magnetic field must be tangent to the plasma-vacuum interface. That this is true for the interior field follows directly from equation (13.6). The condition on the vacuum field is

$$\mathbf{n}\cdot\hat{\mathbf{B}}' = \mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}\times\hat{\mathbf{B}}_0). \qquad (13.22)$$

The second condition follows from the assumption of perfect conductivity: to an observer moving with the interface, the tangential component of the electrical field is zero. Transcribed into the laboratory frame, this condition is

$$\mathbf{n}\times\left[\mathbf{E}' + \frac{1}{c}(\mathbf{v}\times\mathbf{B}_0)\right] = \mathbf{n}\times\left[\hat{\mathbf{E}}' + \frac{1}{c}(\mathbf{v}\times\hat{\mathbf{B}}_0)\right].$$

But

$$\mathbf{E}' + \frac{1}{c}(\mathbf{v}\times\mathbf{B}_0) = 0,$$

therefore

$$\mathbf{n}\times\mathbf{A} = -(\mathbf{n}\cdot\boldsymbol{\xi})\hat{\mathbf{B}}_0. \qquad (13.23)$$

The third boundary condition is that the total pressure, $p + B^2/8\pi$, must be continuous across the boundary, as otherwise infinite acceleration would result. This condition can be written

$$p_0 + \delta p + \frac{(\mathbf{B}_0 + \delta\mathbf{B})^2}{8\pi} = \frac{(\hat{\mathbf{B}}_0 + \delta\hat{\mathbf{B}})^2}{8\pi},$$

together with the equilibrium equations

$$p_0 + \frac{B_0^2}{8\pi} = \frac{\hat{B}_0^2}{8\pi}.$$

Upon subtraction, the result is

$$\delta p + \frac{1}{4\pi}\mathbf{B}_0\cdot\delta\mathbf{B} = \frac{1}{4\pi}\hat{\mathbf{B}}_0\cdot\delta\hat{\mathbf{B}}.$$

In these equations δp means the change in p resulting from the displacement over ξ; the significance of $\delta \mathbf{B}$ is similar. In each case the change is computed moving with the displacement. It follows then that

$$\delta p = p' + \xi \cdot \nabla p$$
$$\delta B = \mathbf{B}' + (\xi \cdot \nabla)\mathbf{B},$$

and when these are used together with equation (13.7) for p' and equation (13.6) or (13.19), for $\mathbf{B}'$ or $\hat{\mathbf{B}}'$, respectively, there is obtained the condition

$$-\gamma p_0 \nabla \cdot \xi + \frac{1}{4\pi}\mathbf{B}_0 \cdot \mathbf{Q} = \frac{1}{4\pi}\mathbf{B}_0 \cdot \nabla \times \mathbf{A} + \frac{1}{8\pi}(\xi \cdot \nabla)[\hat{B}_0^2 - B_0^2]. \quad (13.24)$$

13.54. Returning now to equation (13.17) and setting equation (13.24) into the surface integral, it is found that

$$\delta W - \delta W_f = -\frac{1}{2}\int_S (\mathbf{n}\cdot\xi)\left\{(\xi\cdot\nabla)\left[p_0 + \frac{B_0^2 - \hat{B}_0^2}{8\pi}\right] - \frac{1}{4\pi}\hat{\mathbf{B}}_0\cdot\nabla\times\mathbf{A}\right\} ds.$$

Since both $\mathbf{B}_0$ and $\mathbf{j}_0$ are parallel to the interface, the tangential derivative of $p_0 + \dfrac{B_0^2 - \hat{B}_0^2}{8\pi}$ vanishes, and so

$$\xi\cdot\nabla\left[p_0 + \frac{B_0^2 - \hat{B}_0^2}{8\pi}\right] = (\xi\cdot\mathbf{n})\mathbf{n}\cdot\nabla\left[p_0 + \frac{B_0^2 - \hat{B}_0^2}{8\pi}\right].$$

Furthermore, from the boundary condition equation (13.23),

$$\int_S \frac{1}{4\pi}(\mathbf{n}\cdot\xi)\hat{\mathbf{B}}_0\cdot\nabla\times\mathbf{A}\, ds = -\int_S \frac{1}{4\pi}(\mathbf{n}\times\mathbf{A})\cdot\nabla\times\mathbf{A}\, ds$$
$$= -\int_S \frac{1}{4\pi}\mathbf{n}\cdot(\mathbf{A}\times\nabla\times\mathbf{A})\, ds.$$

Finally, this may be transformed into a volume integral taken over the vacuum region; thus,

$$\int \frac{1}{4\pi}(\mathbf{n}\cdot\xi)\hat{\mathbf{B}}_0\cdot\nabla\times\mathbf{A}\, ds = \int_{\hat{V}} \frac{1}{4\pi}\nabla\cdot[\mathbf{A}\times\nabla\times\mathbf{A}]\, d^3x$$
$$= \int_{\hat{V}} \frac{1}{4\pi}[(\nabla\times\mathbf{A})^2 - \mathbf{A}\times\nabla\times\nabla\times\mathbf{A}]\, d^3x.$$

Combining these results, and making use of equation (13.21), it follows that

$$\delta W = \delta W_f + \delta W_S + \delta W_{\hat{V}},$$

where δW_f is defined by equation (13.18) and

$$\delta W_S = -\frac{1}{2}\int_S (\mathbf{n}\cdot\xi)^2\mathbf{n}\cdot\nabla\left[p_0 + \frac{B_0^2 - \hat{B}_0^2}{8\pi}\right] ds \quad (13.25)$$

and

$$\delta W_{\hat{V}} = \int_{\hat{V}} \frac{1}{8\pi}(\nabla\times\mathbf{A})^2\, d^3x = \int_{\hat{V}}\left[\frac{(\hat{B}')^2}{8\pi}\right] d^3x. \quad (13.26)$$

It will be recalled from §13.21, that the condition for stability of the plasma is that δW be not negative for every ξ which satisfies the boundary conditions.

13.55. By means of the results derived above, it is now possible to determine the conditions of stability of the perfectly diamagnetic plasma confined by a magnetic field. First, it should be noted that if $\mathbf{B}_0 = 0$, then $\mathbf{Q} = 0$ and p_0 is a constant, so that equation (13.18) becomes

$$\delta W_f = \tfrac{1}{2}\int_{\hat{V}} \gamma p_0(\nabla\cdot\xi)^2\, d^3x \geqslant 0. \quad (13.27)$$

In fact, δW_f may be minimized to zero, by choosing ξ divergenceless, without compromising the freedom to choose $(\xi\cdot\mathbf{n})$ arbitrarily.

13.56. Pressure balance requires $\hat{\mathbf{B}}_0^2$ to be constant on the surface; thus, writing

$$\hat{\mathbf{B}}_0 = \hat{B}_0\boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is a unit vector in the $\hat{B}$ direction, so that

$$\nabla(\hat{B}_0^2) = 2\hat{B}_0^2(\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau}.$$

Consequently, equation (13.25) for δW_S may be written as

$$\delta W_S = \frac{\hat{B}_0^2}{8\pi}\int_S (\mathbf{n}\cdot\xi)^2(\mathbf{n}\cdot\mathbf{K})\, ds, \quad (13.28)$$

where $\mathbf{K}$ is the principal curvature vector* defined by

$$\mathbf{K} \equiv (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau}.$$

Because $\hat{B}_0^2$ is constant over the surface, it follows that $\mathbf{K}$ is normal to the surface, and

$$\mathbf{n}\cdot\mathbf{K} = \pm\frac{1}{R},$$

where R is the principal radius of curvature of the $\hat{B}$ line through the point in question, and the sign depends upon whether $\mathbf{K}$ points outward or inward from the surface. It will be apparent that if $\mathbf{K}$ points outward, the integrand of equation (13.28) is everywhere nonnegative. Since $\delta W_{\hat{V}}$ is certainly not negative, the system is stable.

13.57. Suppose, now, that over some region $\mathbf{K}$ points inward. The value of $(\xi\cdot\mathbf{n})$ will be chosen so as to vanish outside the given region. In order to treat this region, it is convenient to introduce orthogonal, curvilinear coordinates (u^1, u^2, u^3), so chosen that $\hat{B}_1 = 0$ everywhere and, further, that $\hat{B}_3 = 0$ on the surface. Thus the surface is a u^1 surface and on it $\hat{\mathbf{B}}$ points in the u^2 direction. Over the region being considered it is postulated that

$$(\mathbf{n}\cdot\xi) = \sin\alpha\chi_2\cdot\sin\beta\chi_3, \quad (13.29)$$

* This is one of the Frenet formulae [31].

where χ_i is defined by

$$\chi_i \equiv \int h_i\, du^i,$$

so that the χ_i are arc lengths along the i curves.

13.58. In order to calculate $\delta W_{\hat{V}}$, it should be noted that, since $\hat{\mathbf{B}}'$ is curl-free, it is possible to write

$$\hat{\mathbf{B}}' = -\nabla\Phi$$

and, therefore,

$$\delta W_{\hat{V}} = \frac{1}{8\pi}\int_{\hat{V}} (\nabla\Phi)^2\, ds.$$

This is minimized by choosing Φ to satisfy the equation

$$\nabla^2\Phi = 0 \tag{13.30}$$

and as a result of this choice it is possible to perform a partial integration which yields

$$\delta W_{\hat{V}} = -\frac{1}{8\pi}\int_S (\mathbf{n}\cdot\nabla\Phi)\Phi\, ds. \tag{13.31}$$

13.59. In the postulated coordinate system, the boundary condition equation (13.22) becomes

$$-(\mathbf{n}\cdot\nabla\Phi) = \frac{h_1}{h_2}\hat{B}_0\partial_2\left(\frac{\xi_1}{h_1}\right) = -\frac{1}{h_1}\partial_1\Phi,$$

so that on the surface, Φ has the form

$$\frac{1}{h_1}\partial_1\Phi = -B_0\alpha\cos\alpha\chi_2\cdot\sin\beta\chi_3 + \Theta. \tag{13.32}$$

Solving the Laplace equation with this boundary condition yields

$$\Phi = B_0\frac{\alpha}{\gamma}\cos\alpha\chi_2\cdot\sin\beta\chi_3\cdot e^{-\gamma\chi_1} + \Theta \tag{13.33}$$

where

$$\gamma^2 = \alpha^2 + \beta^2. \tag{13.34}$$

Here the symbol Θ represents terms of the order of curvature divided by α or β.

13.60. The use of equation (13.29) now yields

$$\delta W_S = -\frac{A\hat{B}^2}{32\pi\overline{R}},$$

where A is the area over which $(\mathbf{n}\cdot\boldsymbol{\xi})$ is not equal to zero and $\overline{R}$ is the average radius of curvature over this area. Similarly, equations (13.32) and (13.33) give

$$\delta W_{\hat{V}} = \frac{A\hat{B}^2}{32\pi}\cdot\frac{\alpha^2}{\gamma} + \Theta,$$

so that

$$\delta W = \frac{A\hat{B}^2}{32\pi}\left[\frac{\alpha^2}{\gamma} - \frac{1}{\overline{R}} + \Theta'\right].$$

Choosing α as small as possible, that is, of the order of the reciprocal length of the region A, and β very large, the terms $\frac{\alpha^2}{\gamma} + \Theta'$ may be made as small as desired. Hence, δW can be made negative and the system is unstable.

APPENDIX B: NORMAL MODE ANALYSIS OF THE PINCH WITH SHARP BOUNDARY

13.61. As an illustration of the application of the normal mode procedure for determining the conditions of stability of a plasma, an idealized model of the cylindrical pinch will be considered. The plasma is assumed to be in the form of a cylindrical rod in the interior of which no current flows. There is, however, a magnetic field in the plasma; hence, in cylindrical coordinates, the interior field has the form

$$\mathbf{B}_0 = (0, 0, B_z); \quad B_z = \text{constant}, \tag{13.35}$$

where, as before, $\mathbf{B}_0$ is the equilibrium field vector. Outside the plasma, the most general field possible is assumed, namely,

$$\hat{\mathbf{B}}_0 = (0, \hat{B}_\theta, \hat{B}_z) \tag{13.36}$$

$$\hat{B}_z = \text{constant} \tag{13.37}$$

$$\hat{B}_\theta = \hat{b}/r, \quad \hat{b} = \text{constant}. \tag{13.38}$$

As in Appendix A, a circumflex over a symbol indicates the value in the vacuum outside the plasma. The plasma pressure is constant, and so the normal mode equation (13.9) reduces to

$$-\rho\Omega^2\boldsymbol{\xi} = \nabla(\gamma p\nabla\cdot\boldsymbol{\xi}) - \frac{1}{4\pi}\mathbf{B}\times(\nabla\times\mathbf{Q}), \tag{13.39}$$

the zero subscripts, indicating equilibrium quantities, having been omitted since this will cause no confusion. The three components of equation (13.39) represent the equations of motion which, together with the appropriate boundary conditions, permit a solution of the problem.

13.62. Because of the linear nature of the equations, the spatial component of the displacement vector may be analyzed into its Fourier components by writing

$$\boldsymbol{\xi} = [\xi_r(r)\xi_\theta(r)\xi_z(r)]\exp(im\theta + ikz). \tag{13.40}$$

When this is done it follows, from the definition of $\mathbf{Q}$ in §13.18, that

$$Q_r = ikB_z\xi_r \tag{13.41}$$

$$Q_\theta = ikB_z\xi_\theta \tag{13.42}$$

$$Q_z = -B_z\left[\frac{1}{r}\partial_r(r\xi_r) + \frac{im}{r}\xi_\theta\right]. \tag{13.43}$$

The individual components of equation (13.39) then give the following set of normal mode equations:

$$-\rho\Omega^2\xi_r = \partial_r(\gamma p \nabla \cdot \xi) - \frac{B_z}{4\pi}(\partial_r Q_z - ikQ_r) \qquad (13.44)$$

$$-\rho\Omega^2\xi_\theta = \frac{im}{r}(\gamma p \nabla \cdot \xi) - \frac{B_z}{4\pi}\left(\frac{im}{r}Q_z - ikQ_\theta\right) \qquad (13.45)$$

$$-\rho\Omega^2\xi_z = ik(\gamma p \nabla \cdot \xi). \qquad (13.46)$$

The form of these equations suggests that the dependent variables be changed by introducing the new function Ψ defined by*

$$\Psi \equiv \gamma p \nabla \cdot \xi - \frac{1}{4\pi}\mathbf{B}\cdot\mathbf{Q}$$

$$= \left(\gamma p + \frac{B_z^2}{4\pi}\right)\left[\frac{1}{r}\partial_r(r\xi_r) + \frac{im}{r}\xi_\theta\right] + ik\gamma p\xi_z. \qquad (13.47)$$

In terms of Ψ, equations (13.44-46) may be rewritten, respectively, as

$$\left(\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)\xi_r = \partial_r\Psi \qquad (13.48)$$

$$\left(\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)\xi_\theta = \frac{im}{r}\Psi \qquad (13.49)$$

$$\left[\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\left(1 + \frac{B_z^2}{4\pi\gamma p}\right)\right]\xi_r = ik\Psi. \qquad (13.50)$$

13.63. The values of ξ_r, ξ_θ, ξ_z given by the equations (13.48-50) may now be set into equation (13.47). The result is a differential equation for Ψ; thus,

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\Psi}{\partial r} - \left(K^2 + \frac{m^2}{r^2}\right)\Psi = 0 \qquad (13.51)$$

where K^2 is defined by

$$K^2 \equiv \frac{\dfrac{k^2B_z^2}{4\pi} - \rho\Omega^2}{\gamma p + \dfrac{B_z^2}{4\pi}}\left\{1 + \frac{k^2\gamma p}{\dfrac{k^2B_z^2}{4\pi} - \rho\Omega^2\left(1 + \dfrac{B_z^2}{4\pi\gamma p}\right)}\right\}. \qquad (13.52)$$

This, together with equations (13.48-50), which now define the ξ_i, is equivalent to the original equations of motion. Equation (13.51) is a form of Bessel's equation for an imaginary argument [32] and the solution is

$$\Psi = I_m(Kr). \qquad (13.53)$$

* The use of this procedure for solving equations 13.44, 13.45, and 13.46 is based on a suggestion made in connection with a similar problem by Reimar Lüst (private communication). It would seem to be applicable in many cases where there is no current flow in the interior of a plasma.

The constant K^2, and therefore Ω^2, is determined by the boundary conditions. For the vacuum field it is possible to write

$$\hat{\mathbf{B}}' = -\nabla\Phi,$$

and, representing Φ by

$$\Phi = \Phi(r)\exp(im\theta + ikz),$$

Laplace's equation takes the form

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\cdot\frac{d\Phi}{dr} - \left(k^2 + \frac{m^2}{r^2}\right)\Phi = 0,$$

so that

$$\Phi = CI_m(kr) + DK_m(kr), \qquad (13.54)$$

where C and D are constants, to be evaluated below.

13.64. It will now be supposed that there is a perfectly conducting rigid wall of radius R_0 located in the vacuum.* At this wall

$$CI_m'(kR_0) + DK_m'(kR_0) = 0, \qquad (13.55)$$

where the prime implies the derivative with respect to the argument. At the plasma-vacuum interface, where $r = R$, the boundary condition given by equation (13.22) becomes

$$k[CI_m'(kR) + DK_m'(kR)] = i\left(k\hat{B}_z + \frac{m}{r}\hat{B}_\theta\right)\xi_r$$

or, using equation (13.48),

$$CI_m'(kR) + DK_m'(kR) = \frac{iK\left(k\hat{B}_z + \dfrac{m}{r}\hat{B}_\theta\right)}{k\left(\dfrac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)}I_m'(kR). \qquad (13.56)$$

From equations (13.55) and (13.56), it follows that

$$C = \frac{K_m'(kR_0)}{I_m'(kR)K_m'(kR_0) - K_m'(kR)I_m'(kR_0)}\cdot\frac{iK\left(k\hat{B}_z + \dfrac{m}{r}\hat{B}_\theta\right)I_m'(KR)}{k\left(\dfrac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)}$$

$$D = \frac{-I_m'(kR_0)}{I_m'(kR)K_m'(kR_0) - K_m'(kR)I_m'(kR_0)}\cdot\frac{iK\left(k\hat{B}_z + \dfrac{m}{r}\hat{B}_\theta\right)I_m'(KR)}{k\left(\dfrac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)}$$

13.65. The boundary condition represented by equation (13.24) can be written as

* In order to avoid possible confusion with the general coordinate r, the radii of the wall, and of the plasma will here be represented by R_0 and R, respectively, instead of the corresponding lower-case symbols used in the main text.

$$-\Psi = \frac{i}{4\pi}\left(k\hat{B}_z + \frac{m}{r}\hat{B}_\theta\right)\Phi + \frac{1}{8\pi}\xi_r\partial_r(\hat{B}_z^2 + \hat{B}_\theta^2),$$

and upon setting in the values for Ψ, Φ, ξ_r, $\hat{B}_\theta$, and the expressions for the constants C and D given above, the result is

$$4\pi\frac{\left(\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)I_m(KR)}{KI_m'(KR)} = \frac{\hat{B}_\theta^2(R)}{R} + \frac{1}{k}\left(k\hat{B}_z + \frac{m}{r}\hat{B}_\theta\right)^2\left[\frac{K_m'(kR_0)I_m(kR) - I_m'(kR_0)K_m(kR)}{K_m'(kR_0)I_m'(kR) - I_m'(kR_0)K_m'(kR)}\right]. \quad (13.57)$$

This is seen to be a dispersion equation which relates the frequency Ω of the waves to the propagation vector defined by

$$\mathbf{k} \equiv \left(0, \frac{m}{r}, k\right).$$

13.66. The dispersion equation is transcendental and can be solved only by numerical means. If the values of Ω^2 are positive then the system is stable, but if they are negative then it must be unstable. The actual determination of which configurations are stable and which are not is laborious, but it has been carried through and the results obtained were given in §13.27 *et seq.*

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